Summetry: A New Word for an Old Arithmetic, Shown on Platonic and Archimedean Solids

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S = mT / r

the sum a shape compels

Abstract. Summetry is symmetry that carries arithmetic: where symmetry says which sets of a figure’s parts are equivalent, summetry requires them to sum alike, and because the groups exhaust a fixed total a known number of times the sum is not chosen but computed. This paper states that relation as $S = mT/r$, names the property, and attributes the underlying identity to the magic labelling literature, where it has been standard since 1963. It then turns the relation on the solids. All five Platonic and all thirteen Archimedean solids carry the same arrangement — twelve things, six groups of four, each thing in two groups — and therefore the same constant, 26, reached through the same 960 numberings; the arrangement is confirmed on several Catalan and Kepler-Poinsot solids as well. Of the twenty-four arrangements of that shape, exactly eight admit a numbering summing to 26, and two of those cannot occur as the faces of any polyhedron. Where the memberships are not uniform the constant is no longer determined but confined to an interval, the Summetry Range, bounded by the least and greatest weights W– and W+; the Formula is the Range with zero width. The Tabakgasse polyhedron of Budapest, which refuses summetry at three successive stages, is given as a counterexample.

summetry /ˈsʌmɪtri/ n. (pl. summetries) [blend of sum + symmetry, coined by G. Földvári, 2026]
1. The property of a figure whose symmetry compels a constant sum: a numbering of its parts such that every group of parts related by that symmetry adds to the same total. 2. That total itself; the value S in the relation $S = mT/r$.

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The word

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summetric adj. (also summetrical) · summetrically adv. · summetrise v.t. (US summetrize), to calculate the summetry of a figure · summetrisation n.

Hungarian: szummetria fn., szummetrikus mn., szummetrikusan hsz., szummetrizál ige.

Note. Summetria is also a standard transliteration of the Greek συμμετρία (summetria), due proportion. The blend is meant to recall it.

In plain words

Write numbers on the parts of a shape: on its corners, or its edges, or its faces. Then gather those parts into groups the shape itself provides, such as the four edges around each face. Ask that every group add up to the same total. That total is not a matter of choice. Count the parts, count the groups, and count how many groups each part belongs to, and the total is already decided. This paper writes that in one line, S = mT/r, and calls the property summetry.

The arithmetic is not new; it has been known to mathematicians since 1963 under the name magic labelling, and that work is cited throughout. What is new is what the arithmetic finds when it is turned on the solids. All five Platonic solids and all thirteen Archimedean solids carry the same answer, 26, arriving in the same way: twelve things, six groups of four, each thing in two groups. They are not eighteen coincidences. They are one pattern in eighteen costumes, and the same 960 numberings serve every one of them.

1. The formula, and where it comes from

Two words first. A part is one of the numbered things on the shape: a corner, an edge, a face, or whatever else the shape offers. A group is a set of parts that the shape brings together, such as the four edges around one face. The size of a group is how many parts it contains.

Everything rests on one observation. Add up all the groups. Each part is counted once for every group that contains it. So the grand total is not the sum of the numbers, but the sum of the numbers with each one repeated according to how many groups it belongs to.

If every group is to show the same total, that grand total must divide evenly among the groups. Written out:

The Summetry Formula

$$S = mT / r$$

- **S** the sum every group must show
- **m** how many groups each numbered part belongs to
- **T** the total of all the numbers used, that is 1 + 2 + 3 and so on to the last
- **r** how many groups there are, and so how many times the sum repeats

S, m, T and r are all positive whole numbers, none of them zero.

Read aloud: total, times membership, shared out over the repeats.

The gate

S must come out a whole number. If mT/r is not whole, no arrangement exists at all, and this is proved rather than merely unfound. Gathering the tetrahedron’s six edges into its four faces gives 2 × 21 / 4, which is ten and a half. Barred. Gathering the cuboctahedron’s twelve vertices into its eight triangles gives 2 × 78 / 8, nineteen and a half. Also barred — on the very solid whose six squares carry 26.

A second gate: groups of two

No part may belong to two different groups of size two. Suppose some part x belonged both to the group holding x and a, and to the group holding x and b, with a and b different parts. Each group must total S, so the number on a would equal S minus the number on x, and so would the number on b. The two would carry the same number, which cannot happen when every part carries a different one. This single line accounts for every failure in the tables below.

Attribution

The identity is classical. It is the standard opening move of magic labelling theory, which begins with Sedláček in 1963 [1] and continues through Kotzig and Rosa [2], MacDougall, Miller, Slamin and Wallis [3], and the surveys of Wallis [4] and Gallian [5]. The counting step appears there verbatim: the sum of all the group weights counts each label exactly as often as its multiplicity [6]. The non-integer test is stated in the same literature [4]. No claim of priority is made here for the relation. It is restated in the compact form $S = mT/r$, which will serve throughout, and the property is given a name.

A summetric numbering is a magic labelling in the established sense. Summetry names the property that forces the constant, and the formula is its shorthand.

The field is still moving

This is not a closed chapter. A 2026 paper by Kovář, Rozman and Šparl [11] works on distance magic graphs, where the numbers sit on the vertices and each group is the set of one vertex's neighbours. In the notation used here, m is the degree and r the number of vertices. What they are after is labellings they call self-reverse: ones that survive a reversal of the labels and so agree with the graph's own symmetry, which lets the graph and its labelling be described together by a smaller quotient. They give a construction that builds a new distance magic graph from two old ones, and classify every connected four-valent example up to order thirty.

That concern is close to the one named here. Their interest is not merely that the sums come out equal, but that the labelling respects the symmetry rather than sitting on it by accident — which is what summetry means. The reversal itself has a familiar look: sending a label to its complement is the same move as the value complementation of Star Face, where v is sent to $13 - v$.

2. From the flat star to the solid

A magic star with n points carries 2n numbers, one at each tip and each crossing, arranged on n straight lines of four. Every number sits on exactly two of those lines. Feed that into the formula.

$$S = 2 \cdot n(2n + 1) / n = 4n + 2$$

T the total, 1 + 2 + ... + 2n, which is n(2n + 1)
m 2, because each number lies on two lines
r n, the number of lines

So a five-pointed star would have to make 22, a six-pointed star 26, a seven-pointed star 30. The star does not choose; the shape does. This is the classical magic-star constant, and it is a special case of the same identity.

The five-pointed star is worth pausing on, because it shows exactly how far the formula reaches. It gives 22, and 22 is a whole number, so the gate opens. And yet no such star exists: there is no way to place 1 to 10 on a pentagram so that all five lines make 22, as an exhaustive check of the arrangements confirms. The formula says what the sum must be if a star exists. It does not promise one.

The hexagram is the first magic star that can be built, and its constant is 26.

The three-dimensional test

The six-pointed star matters here, because 4n + 2 at n = 6 gives 26 — and that star is not merely a picture. It lives inside a solid. Escher's Solid, the stellated rhombic dodecahedron, has eight inner apexes. Face any one of them and the outline becomes a magic hexagram: twelve of its triangular faces, six lines of four. The flat star has been lifted into three dimensions.

The question Star Face [7] asks is whether the constant survives the lift. It does, and more than once. Reading the solid's twelve diametric quadruples gives 26. Reading its twelve four-faced pyramids gives 26 again, from a different partition at a different level.

The flat star gives 26 once. The solid gives it again, twelve times, on the total of 48 triangular faces meeting in the twelve apexes of its pyramid stellations.

And the two are the same object, not merely the same number. Twelve things, six groups of four, every thing in two groups: that is the hexagram's pattern, and it is the pattern this paper finds on all eighteen solids.

A footnote worth checking. The classical count of essentially different magic hexagrams is 80, and the hexagram has 12 symmetries. $80 \times 12 = 960$ — the same 960 that appears throughout what follows.

3. Twelve things, six groups

Take twelve things, six groups of four, and let every thing belong to exactly two groups. Then $T = 78$, $m = 2$, $r = 6$, and the formula gives 26 with no further information required.

One arrangement, and only one

Suppose in addition that no two groups share more than one thing. Then there is exactly one way to build the structure, and the proof is short enough to read aloud.

Each thing lies in two groups, so name it by that pair: the one in A and C. If two groups shared two things, those two things would have the same name, which the extra condition forbids. Six groups make fifteen possible pairs. Twelve are used and three are left empty. So each group meets four of the other five and misses exactly one. Pair each group with the one group it misses. Those two groups share no thing at all, and the six groups therefore fall into three such pairs, with no group left over. Call them the first, second and third pair.

Now count the things again from the other end. A thing lies in two groups, and those two cannot come from the same pair, since a pair shares nothing. So a thing chooses two of the three pairs, then one group out of each: three ways to choose the pairs, two ways within the first, two within the second. That is $3 \times 2 \times 2 = 12$, exactly the number of things, with nothing left over. Nothing was chosen anywhere. The arrangement is forced.

Its natural home is the octahedron: the six groups are its vertices, the twelve things its edges. Every other appearance is that same object, dualised or subdivided.

4. The eighteen Platonic and Archimedean solids

Each solid below was built from coordinates, its rotation group computed from its own vertices, and its features found by symmetry orbit. Nothing about orientation was assumed. In every line m = 2, T = 78, r = 6, and so S = 26.

solid	the twelve	the six
tetrahedron	face-corners	edges
cube	edges	faces
octahedron	edges	vertices
dodecahedron	faces	inscribed cube
icosahedron	vertices	inscribed cube
truncated tetrahedron	hidden rectangles	edges of the parent
cuboctahedron	vertices	squares
truncated cube	faces	faces
truncated octahedron	edges	faces
rhombicuboctahedron	faces	faces
truncated cuboctahedron	edges	faces
snub cube	edges	faces
icosidodecahedron	faces	inscribed cube
truncated dodecahedron	faces	inscribed cube
truncated icosahedron	faces	inscribed cube
rhombicosidodecahedron	faces	inscribed cube
truncated icosidodecahedron	faces	inscribed cube
snub dodecahedron	faces	inscribed cube

The same fingerprint every time — three group-pairs sharing nothing, twelve sharing exactly one — and the same 960 numberings. A solution found on any one can be carried to the other seventeen.

The truncated tetrahedron’s twelve are hidden

For each edge of the parent tetrahedron, take the four vertices lying on the edges that leave that edge’s endpoints without running along it. All six quadruples are coplanar: six invisible rectangles inside the solid.

Beyond the eighteen

The same design, and the same 960, were confirmed on the rhombic dodecahedron, the rhombic triacontahedron, the triakis tetrahedron, the great dodecahedron and the small stellated dodecahedron. The Catalan solids and the Kepler-Poinsot star polyhedra inherit it.

5. Why it cannot fail: the point groups

Every Archimedean solid inherits the rotation group of a Platonic one, and each of those groups carries the same skeleton. The names below are the standard Schoenflies labels used throughout this field.

group	order	solids
T_d	24	tetrahedron, truncated tetrahedron
O_h	48	cube, octahedron, cuboctahedron, the truncations, rhombic dodecahedron
O	24	snub cube — chiral, rotations only
I_h	120	dodecahedron, icosahedron, icosidodecahedron and the rest
I	60	snub dodecahedron — chiral
D_6h	24	hexagonal prism
D_3h	12	triangular orthobicupola, J27

The cube and octahedron are O_h, of order 48, with rotation subgroup O, of order 24. It has thirteen axes: three fourfold, four threefold, six twofold. The six twofold axes give twelve elements, the three fourfold axes give six groups. That is the cube's edges and faces, whatever the solid is wearing. For the icosahedral solids, six fivefold axes give twelve faces of the pentagonal class, and the six groups come from an inscribed cube. The icosahedral group has fifteen twofold axes forming exactly five orthogonal triples — the five inscribed cubes — and all five give the same design, so the choice among them is harmless.

What decides is transitivity, not handedness

Two solids make the point sharply. The triangular gyrobicupola is the cuboctahedron, group O_h, and it carries 26 with 960 numberings. The triangular orthobicupola is J27, group D_3h of order 12, and it carries none. Same twelve vertices, same six squares, same m = 2, same S = 26 by the formula. Give one cupola a sixth of a turn and the numbering vanishes, because the smaller group no longer moves all twelve vertices alike.

Neither solid is chiral, so handedness is not the discriminator. Genuine chirality appears elsewhere: the snub cube and the snub dodecahedron have no mirrors at all, and both carry 26 without difficulty.

6. Which arrangements can carry 26

Drop the condition that two groups share at most one thing, and the family grows. Twelve things, six groups of four, each thing in two groups, is exactly a 4-regular multigraph on six nodes: the nodes are the groups and the edges are the things. There are twenty-four such arrangements. Exactly eight admit a numbering that makes every group sum to 26.

arrangement	numberings	as faces of a solid
three fourfold shares	2,654,208	impossible
one fourfold share	313,344	impossible
six doubles, one ring	64,512	hexagonal prism
six doubles, two triples	46,080	prism–antiprism–prism stack
four doubles	24,960	arrangement exists, solid unnamed
three doubles	11,040	no planar arrangement found
two doubles	5,760	arrangement exists, solid unnamed
all single — the octahedron	960	the eighteen, and more

Two are impossible for a reason worth stating. In any polyhedron two faces meet in at most an edge, so at most two vertices; the top two arrangements demand four. The first of the two, the one carrying 2,654,208 numberings, is Star Face's own star: three diametric quadruples each counted twice. Its doubling comes from counting, not from geometry, so it could never have been a solid.

One caution. The hexagonal prism and the prism-antiprism-prism stack share an identical overlap profile and yet give 64,512 and 46,080. Six doubles arranged as one ring, or as two triangles. The profile is not a complete invariant: the arrangement must be named, not merely described.

No prior statement of these counts has been found. The 960 has a published neighbour, being the classical count of 80 essentially different magic hexagrams multiplied by the hexagram's twelve symmetries.

7. When the sum is not forced: the Summetry Range

Everything so far assumed that every part belongs to the same number of groups. Drop that, and the sum stops being decided.

Number all twenty-six elements of a cube from one pool, 1 to 26, and let each face take its own number, its four edges and its four corners. A face belongs to one group, an edge to two, a corner to three. Adding up the groups now gives

$$6S = 351 + E + 2V$$

E the total of the numbers that land on edges
V the total of the numbers that land on corners
351 the total of 1 to 26, which is T

S now depends on which numbers land where. Put the smallest numbers where they count most and you reach the floor; put the largest there and you reach the ceiling. Write W for the weight of a numbering, meaning the sum of every number multiplied by its own membership. Then

The Summetry Range Formula

$$\lceil W- / r \rceil \quad \leq \quad S \quad \leq \quad \lfloor W+ / r \rfloor$$

W– the least weight: largest memberships paired with smallest numbers
W+ the greatest weight: largest memberships paired with largest numbers

This follows from the rearrangement inequality, and for the cube it gives 100 to 143 — forty-four values, every one attainable. The same shape and the same numbers: uniform membership gives one answer, mixed membership gives forty-four.

When every membership is equal the two lists cannot be rearranged against one another, the two weights coincide at mT, and the range collapses to a point.

The Summetry Formula
is the Summetry Range with zero width.

$$S = mT / r$$

Attribution, again

This range is known in the literature as the spectrum of a magic labelling problem, and the bounds are computed in the same way by MacDougall, Miller, Slamin and Wallis [3] and in the survey literature [5]. The name Summetry Range is adopted for it here; no claim is made beyond the name.

What is confirmed and what is open

Every value in the range has been realised for the tetrahedron, cube, octahedron and dodecahedron. For the icosahedron, 107 of the 132 values carry a certificate and 25, all at the low end, remain undecided. None has been shown impossible. Whether the bound is sharp there is open.

8. A solid that refuses, three times over

A formula that only ever confirms is a coincidence, not a formula. The Tabakgasse polyhedron [8], the lantern solid of the Dohány Street Synagogue in Budapest, refuses summetry at every stage, and never twice for the same reason.

Its core has 16 vertices, 28 edges and 14 faces: two squares, four hexagons and eight triangles. Raising a pyramid on twelve of those faces gives twelve pyramids, the same count as Escher's Solid.

First refusal: symmetry

The full symmetry group has order 16 and splits the twelve pyramids into orbits of four and eight. No arrangement of six groups of four survives the symmetry. Twelve pyramids are necessary but not sufficient; they must also be interchangeable.

Second refusal: arithmetic

Number the 28 edges and group them by the 14 faces. Each edge lies on two faces, so $m = 2$, $T = 406$, $r = 14$ and $S = 58$, a whole number. The gate opens, and yet it is impossible. Write P, Q, R for the sums of the four triangle-triangle edges, the sixteen hexagon-triangle edges and the eight square-hexagon edges. Then $P + Q + R = 406$. The two squares are edge-disjoint and each sums to 58, so $R = 116$. Adding the eight triangles counts P twice and Q once, giving $2P + Q = 464$. Subtracting, $P = 174$. But P is the sum of four numbers from 1 to 28, and the largest four make 106.

Third refusal: divisibility

Raise pyramids on all fourteen faces and the solid has 30 vertices, 84 edges and 56 faces. Grouping the edges by the faces gives $2 \times 3570 / 56$, which is 127.5. The gate shuts.

9. Related work by the author

Star Face [7] is a sieve model demonstrated on Escher's Solid, descending from a population well over the number of atoms in the Solar System to 6,223,974 magic numberings, and juxtaposing the readings of the solid's twelve pyramids with masking in medical diagnosis. It is where summetry was first named.

The GooglyCube [9] carries a single arrangement in which many shapes agree at once: every face makes 26 on its edges and 18 on its corners, so 44 together, and the hidden cuts through the body — hexagons, tetrahedra, symmetry planes — return 26 as well. The cube's twenty-six marked points, six face centres, twelve edge midpoints and eight corners, are the same twenty-six the faces have to make.

The VIO formula [9] is a separate observation on the cube: its edge times its body-diagonal, divided by its face-diagonal, equals the diameter of the largest circle that fits inside it. For unit edge that is the square root of 3 over the square root of 2, or about 1.2247.

The Tabakgasse polyhedron [8] is treated in section 8 above, and in Star Face at folios 79 and 80.

10. Open questions

Named solids for the four-double and the two-double arrangements, which carry 24,960 and 5,760 numberings. The quadrilaterals of each can be laid out in the plane, but neither has yet been found on a solid.

The three-double arrangement, which carries 11,040 numberings and whose quadrilaterals cannot be laid out in the plane under any cyclic ordering. The search was exhaustive for spherical solids but not for toroidal ones. If none exists, arithmetic permits what geometry refuses — a theorem worth having either way.

Whether every value in the Summetry Range is attained on the icosahedron.

Whether the eight numbering counts have a closed form

Each of them — 960, 5,760, 11,040, 24,960, 46,080, 64,512, 313,344, 2,654,208 — is known only because a computer counted the numberings one by one. A closed form would give the count straight from the arrangement, as S = mT/r gives the sum without trying arrangements.

Two words are needed to say what has been found. An arrangement can sometimes be shuffled without changing it: relabel the twelve things among themselves so that every group still contains exactly the things it did before. Such a shuffle is called an automorphism, and the number of them is written |Aut|. The octahedron arrangement, for instance, has 48 of them. Now take one numbering and apply all those shuffles to it. What comes out is a family of numberings that are the same solution seen from different sides. That family is called an orbit, and the number of orbits is the number of essentially different numberings.

For all eight arrangements the count divides exactly by the size of that arrangement's automorphism group:

arrangement	numberings	|Aut|	orbits
three fourfold shares	2,654,208	663,552	4
one fourfold share	313,344	6,144	51
six doubles, one ring	64,512	768	84
six doubles, two triples	46,080	4,608	10
four doubles	24,960	128	195
three doubles	11,040	96	115
two doubles	5,760	64	90
all single	960	48	20

So the numberings are not arbitrary: they factor through the symmetry of the arrangement, as numberings equals |Aut| times orbits. The first half of that product is easy, since group orders can be computed. The second half is the question that remains, and a formula for the orbit counts would give a closed form for all eight.

No pattern has been found in the orbit counts — 4, 51, 84, 10, 195, 115, 90 and 20 — and none is claimed here to be absent. The question is simply open.

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