Race Outcome Formula

Versenykimenetel képlet · bilingual edition

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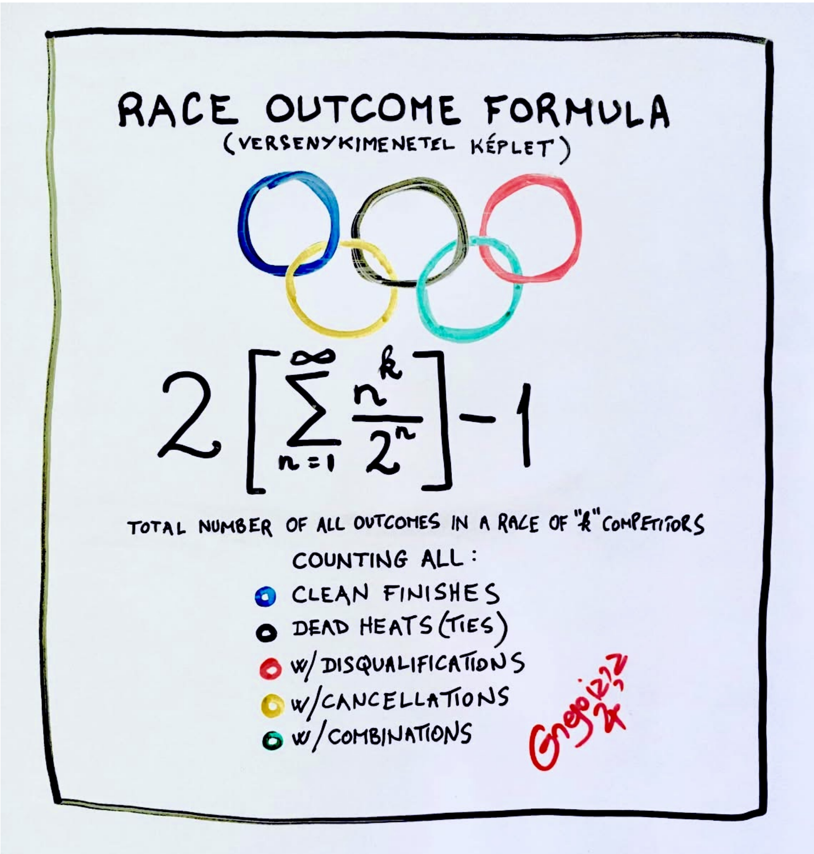
Author's note. A race is not only won. It is also tied, abandoned, and never started. This paper counts every way it can end, and argues that the difference between leaving before the starting pistol and being removed after it is worth an entire term of the arithmetic.



The author's drawing, 10 July 2024, made for the centennial of the Paris Olympic Games and linked from the Online Encyclopedia of Integer Sequences at A007047.

Abstract. How many ways can a race end? If everyone finishes and nobody ties, the answer is k factorial. Allow dead heats and the answer becomes the Fubini numbers, which is textbook. But a race also loses competitors: some are disqualified after the start, some never start at all, and those two are not the same event. Counting all of it gives 1, 3, 11, 51, 299, 2163 — eleven outcomes for two competitors, 299 for four, and more than 112 billion for twelve. The sequence was already known as the number of chains in a power set; this paper gives it a second meaning, sets out the model in full, and argues that the distinction between disqualification and withdrawal, which every sporting authority makes and no combinatorial account had, is exactly what the arithmetic is for.

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1 The formula

For a race with **k** registered competitors, the total number of possible outcomes is

$$2 \cdot \left[\sum_{n=1}^{\infty} \frac{n^k}{2^n} \right] - 1$$

where n runs through the positive integers. The bracket is an infinite sum but it always closes on a whole number. For two competitors it gives 6, and the formula gives eleven. For three the bracket is exactly **26** and the formula gives 51. For four it gives 299.

Competitors k	2	3	4	5	6	8	12
Clean finishes only, k!	2	6	24	120	720	40,320	479,001,600
All outcomes	**11**	**51**	**299**	**2,163**	**18,731**	**2,183,339**	**112,366,270,379**

Table 1. Life is more complicated than a finishing order. With twelve registered competitors there are under half a billion clean finishes and over 112 billion outcomes altogether.

2 What the sequence already was

These numbers were catalogued long before I met them, as **A007047** in the Online Encyclopedia of Integer Sequences, under the description *number of chains in the power set of an n-set*. That phrase deserves unpacking, because it sounds forbidding and is not.

Take two things, A and B. Write down every selection you could make from them: nothing at all, just A, just B, or both. That is four selections, and together they are called the **power set**.

Now stack the selections inside one another, like nested boxes or a set of Russian dolls, each fitting within the next. ∅ ⊂ {A} ⊂ {A,B} is a stack of three — nothing, then A, then both. {B} ⊂ {A,B} is a stack of two. A single selection on its own counts as a stack of one. Such a stack is a **chain**.

Count every chain you can build from two things and there are eleven. From three things, 51. From four, 299. That is the whole of what the description means.

Which is where the surprise lies, because eleven is also the number of ways a two-horse race can end. The two lists are the same length, and they are the same list.

#	The eleven chains	The eleven race outcomes
1	`∅`	`AfBf`
2	`∅ ⊂ {A}`	`BfAf`
3	`∅ ⊂ {A} ⊂ {A,B}`	`AtBt`
4	`∅ ⊂ {B}`	`AfBd`
5	`∅ ⊂ {B} ⊂ {A,B}`	`AfBc`
6	`∅ ⊂ {A,B}`	`BfAd`
7	`{A}`	`BfAc`
8	`{A} ⊂ {A,B}`	`AdBd`
9	`{B}`	`AcBc`
10	`{B} ⊂ {A,B}`	`AdBc`
11	`{A,B}`	`AcBd`

Table 2. Left, the chains in the power set of {A, B}, written by mathematicians thinking about nested sets. Right, the outcomes of a race between A and B: **f** finished, **t** tied, **d** disqualified, **c** cancelled. The order of the two columns is not meaningful; the count is.

3 The eleven, in full

Two competitors, A and B. Each may finish, and if both finish they may tie. Each may instead be disqualified after the start, or cancel before it.

Group	Outcomes	Count
Both finish, no tie	`AfBf, BfAf`	2
Both finish, dead heat	`AtBt`	1
One finishes, one does not	`AfBd, AfBc, BfAd, BfAc`	4
Neither finishes	`AdBd, AcBc, AdBc, AcBd`	4
Total		**11**

Table 3. The eleven outcomes for two competitors, as submitted to the Online Encyclopedia of Integer Sequences and accepted on 28 July 2024.

It is worth walking the count up slowly, because the arithmetic is easy to get wrong. Three outcomes if both finish: A first, B first, or a dead heat. Add the cases where exactly one finishes and the other is disqualified or cancels, which is four more, making seven. Add the cases where neither finishes, each independently disqualified or cancelled, which is four more again. Eleven. The original account of 2024 walked the same total up as 3, 6, 9, 11, grouping the cases differently; the count is the same.

4 The model, term by term

The formula of section 1 is compact but it hides the reasoning. Splitting the count by how many competitors actually finish makes every part of the model visible:

$$a(k) \;=\; \Sigma_j \; C(k, j) \;\cdot\; \text{Fubini}(j) \;\cdot\; 2^{k-j}$$

summed over j = 0, 1, 2, … , k. Every symbol, in full:

Symbol	Meaning
a(k)	The answer: the total number of possible outcomes of a race for which k competitors have registered.
k	How many competitors registered — registered, not started. Some of them may never reach the line.
$\boldsymbol{\Sigma_j}$	The summation sign. Add up one term for each value of j.
j	The number who actually finish. It runs from 0, nobody finishing, to k, everybody finishing. Each value of j is a separate case and the cases cannot overlap, so adding them counts nothing twice.
C(k, j)	“k choose j”: the number of ways to pick *which* j of the k competitors are the finishers. C(4,2) = 6, because there are six ways to choose two people out of four. A choice of who, carrying no order.
Fubini(j)	The number of ways j finishers can be ranked when ties are allowed. Were ties forbidden this would be j factorial, but two runners may cross together, so it is larger. The values run 1, 1, 3, 13, 75, 541, 4683. Fubini(2) = 3: A ahead of B, B ahead of A, or a dead heat. This factor carries both the ranking and the ties.
$\boldsymbol{2^{k-j}}$	The fates of the non-finishers. There are k − j of them, and each independently is either **disqualified** after the start or **cancelled** before it. Two possibilities each, so 2 multiplied by itself k − j times. This is where the argument of section 6 lives: the base is 2 and not 1 precisely because those two absences are different events.

Table 4. The formula glossed in full.

In one sentence: **choose who finishes, rank them allowing ties, and give everyone else one of two fates — then add up over every possible number of finishers.**

Worked out for two competitors, the three lines of the sum are exactly the three groups of Table 3:

j finishers	C(2, j)	Fubini(j)	2^{2-j}	Term
0	1	1	4	**4**
1	2	1	2	**4**
2	1	3	1	**3**
			total	**11**

Table 5. The sum for k = 2. For three competitors the terms are 8 + 12 + 18 + 13 = 51; for four, 16 + 32 + 72 + 104 + 75 = 299.

5 The Fubini numbers

The middle factor deserves its own section, both for what it counts and for whose name it carries.

The **Fubini numbers** count the ways a set of things can be ranked when ties are allowed: 1, 1, 3, 13, 75, 541, 4683, 47293, 545835. Three competitors give thirteen finishing orders — six with no tie at all, six with one pair tied, and one with all three abreast.

They were studied by **Arthur Cayley** in 1859, counting a family of trees, and by **William Allen Whitworth** in *Choice and Chance*, where they appear as *preferential arrangements*: the ways a voter might rank candidates with equal preferences permitted. **O. A. Gross** gave them their modern treatment in the *American Mathematical Monthly* in 1962.

They carry two names. **Ordered Bell numbers**, after Eric Temple Bell, whose Bell numbers count the partitions of a set: these count partitions that have also been put in order. And **Fubini numbers**, after Guido Fubini, because they count the equivalent forms a multiple integral can take when the order of integration is rearranged — the subject of Fubini's theorem.

Guido Fubini was born in Venice on 19 January 1879, the son of a mathematics teacher. He studied at the Scuola Normale Superiore in Pisa under Dini and Bianchi, and held chairs at Catania, Genoa, and from 1908 at Turin, teaching at the Politecnico and the University both. In 1938 the Fascist racial laws forced him from his chair because he was Jewish. The Institute for Advanced Study invited him to Princeton the following year and he emigrated with his family, already in poor health. He taught a few more years in New York and died there on 6 June 1943, five years after leaving Italy. His name is on the theorem, on the Fubini–Study metric, and on these numbers.

The race reading of the Fubini numbers is not new, and I claim none of it. The standard illustration in the literature is a horse race with dead heats, and the Encyclopedia's own description of A000670 mentions competitors ranking. What follows is the step beyond it.

Fubini counts the race that sport imagines, in which everyone finishes. A007047 counts the race as it actually happens.

6 Why the distinction is the whole point

That step is made by one decision, and everything else follows from it: **a competitor who does not finish has two possible histories, not one.**

A **cancellation** happens before the start. Injury, illness, a ban, a missed flight, a withdrawal on the morning. The competitor never crossed the line, produced no time, and left no trace in the race itself. Nothing happened.

A **disqualification** happens after the start. The competitor ran. And here is the argument that no combinatorial account had made: **something may have been achieved before the disqualification.** A record split time may stand on the clock. A movement, a reaction, a piece of conduct may have occurred that matters to the athlete's own account of themselves, to the standing of that particular race or heat, to the inquiry that follows, or to the sport at large. The result is void. The event is not.

Sport has always known this even where mathematics did not. A false start and a failed test both erase a result, but only one of them leaves a performance behind to be argued over. A rider disqualified for interference in the final furlong still rode the first six. A relay squad thrown out for a bad changeover still ran the legs. The record books may refuse the time; the stopwatch recorded it, and the inquiry will want it.

There is a matter of fairness in it too. A disqualification generally implies some fault on the competitor's part. A cancellation generally implies the opposite: an injury, an illness, an accident of scheduling, something that happened to them rather than something they did. To count the two as one outcome is to say that being thrown out and being unable to appear are the same fate. They are not, and no athlete would accept that they were.

That single asymmetry is what turns the thirteen of Fubini into the fifty-one of A007047. It is the **2** in $\mathbf{2^{k-j}}$. Set that 2 to 1 — treat every absence as the same absence — and the count collapses back to a race in which nobody is ever removed, only missing.

The arithmetic makes the point more sharply than any argument can. Counted with absence undifferentiated, the outcomes run 1, 2, 6, 26, 150, 1082. Counted with the distinction restored, they run 1, 3, 11, 51, 299, 2163. Term for term, the second is exactly twice the first, less one.

Registered competitors	Absence undifferentiated	With the distinction
2	6	**11** = 2 · 6 − 1
3	26	**51** = 2 · 26 − 1
4	150	**299** = 2 · 150 − 1
5	1,082	**2,163** = 2 · 1,082 − 1

Table 6. Distinguishing the two absences doubles the number of outcomes, less one.

So the distinction does not refine the count. It doubles it. And the doubling is the 2 standing in front of the bracket in the formula of section 1, which has been carrying this argument on its face all along. The distinction is not a refinement of the model. It is the model.

Fubini counts the race that sport imagines, in which everyone finishes. A007047 counts the race as it actually happens.

7 What is claimed, and what is not

Not claimed. The closed form is not new. Benoît Cloitre gave $a(n) = \Sigma_{k\geq 1} (k+1)^n/2^k$ on the Encyclopedia's entry in September 2002; substituting $m = k + 1$ turns it into the formula of section 1 exactly, and the two agree at every value. Nor is the sequence itself mine: it is Sloane's and Nelsen's, and the chains-in-a-power-set reading goes back to Nelsen and Schmidt in 1991. Nor, as section 5 says, is the race reading of the Fubini numbers new.

Claimed. The interpretation: that A007047 counts the outcomes of a race, which no one had noticed. The model that produces it, in which a non-finisher has two histories rather than one. The argument of section 6 for why that distinction is worth making, drawn from what actually happens in sport rather than from the arithmetic. And the worked breakdown of the eleven outcomes for two competitors, which is what makes the correspondence checkable by anyone.

The comment and example were submitted to the Online Encyclopedia of Integer Sequences and accepted on 28 July 2024. The drawing above is linked from the entry. The comment as lodged there says *predictable* outcomes where this paper says *possible*; nothing here predicts anything, and the later word is the better one.

Versenykimenetel képlet

A 2024. július 10-én, a centenáriumi párizsi olimpiára készült eredeti írás.
A közelgő centenáriumi Párizsi Olimpiai Játékok tiszteletére bemutatom azt a matematikai képletet, amely megadja egy verseny összes lehetséges kimenetelének számát k számú nevezett versenyző esetén. A képlet figyelembe veszi, megkülönbözteti és összeadja az alábbi kimenetelek valamennyi kombinációját: sima rajt–cél eredmények; holtversenyek; kizárások rajt után; távolmaradások és eltiltások rajt előtt; végül azok az esetek, amikor egyszerre van kizárás és távolmaradás is.

Vegyük azt az egyszerű esetet, amikor adott két nevezett versenyző, „A” és „B”. Ha A nyer és B második, vagy fordítva, az két lehetőség; a harmadik a holtversenyük. Ehhez jön, ha az egyiket vagy mindkettőt a rajt után kizárják, illetve ha egyik vagy másik, netalán mindkettő távol marad valamiért — például sérülés vagy eltiltás okán. Így az összes lehetséges kimenetel száma tizenegy.

Matematikailag kifejezve a „sima”, egymástól eltérő eredményekkel záruló kimenetelek száma k indulónál k!, azaz k faktoriális. Ugyanakkor mindannyian tudjuk, hogy az élet annál bizony „végtelenszer”, de legalábbis „hatványozottan” bonyolultabb: négy nevezett versenyző esetén az összes lehetséges kimenetel száma 299, nyolc embernél már 2 183 339, egy tucatnál pedig 112 milliárd felett jár.

Fontos megjegyezni, hogy a rajt utáni — tehát inkább a versenyzőnek felróható — kizárás, illetve a rajt előtti körülmények folytán — tehát inkább a nevezett ártatlansága mellett — bekövetkezett távolmaradás megkülönböztetése azért lényeges, mert a rajtot követően kizárt versenyző teljesítménye — például rekord részideje —, a verseny során tanúsított valamely rendhagyó magatartása, mozdulata vagy reakciója mind az egyéni teljesítmény, mind az adott verseny, adott esetben a kizárás utólagos vizsgálata, illetve általánosságban maga az érintett sportág egésze szempontjából is kiemelt jelentőséggel bírhat. E finom megkülönböztetés a képlet egyik kulcsa.

Hatalmas megtiszteltetés számomra, hogy ezt publikálta a világ egyik legnagyobb matematikai adatbázisa, az On-Line Encyclopedia of Integer Sequences: oeis.org/A007047

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Grego — Gergely Földvári, Budapest, 4 September 2026.