COEFFICIAL DIVISIBILITY
- Multimorphism, Transdimension and Singularity

by Grego Gergely Földvári

"Exceptio probat regulam (de rebus non exceptis)"

My intention is to provide fresh ideas and a clearcut approach towards a proof for a famous "folklore" problem in mathematics (congruent to the case of integer 2 in Pillai's Conjecture), namely and simply put as ***"26 is the only integer directly sandwiched between perfect powers"***, while introducing the perhaps novel concepts of Coefficial Divisibility, Multimorphism and Transdimension, all in the context of Singular Incidence.

The methods outlined here can be a kind of complement to research on Catalan's Conjecture (a.k.a. Mihăilescu's Theorem), while the juxtaposition of the concepts of singular incidence, consecutive perfect powers of uniform parity, binomial theorem, Pascal's Triangle, coefficial divisibility, integer multimorphism and transdimensionality may hold valuable points towards further partial (or full) proof attempts, or significantly simplifying existing proofs related to both Catalan's Conjecture and Pillai's Conjecture in number theory.

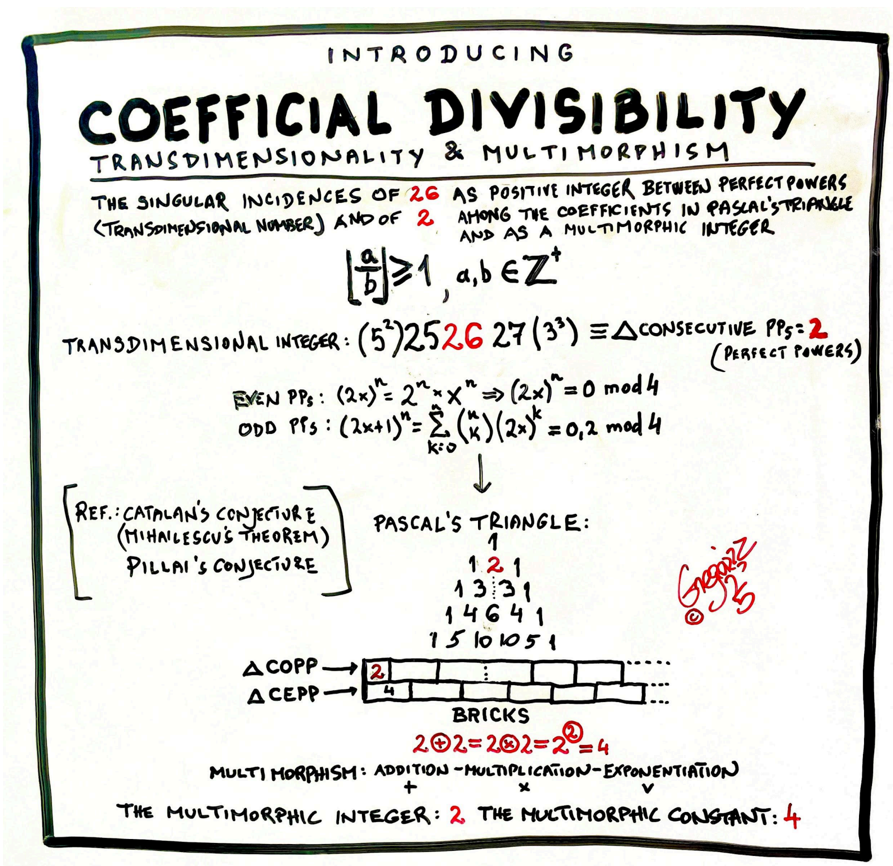

Illustration: photo of my 'maths graffiti' on home whiteboard, source:
https://facebook.com/gergely.grego.foldvari

contact: gfoldvari@alumni.berklee.edu
gfoldvari@gmail.com

link: tinyurl.com/gregos26

DOI: https://doi.org/10.5281/zenodo.22259776

OUTLINE

1. Restatement of the "Sandwiched 26 Singular Incidence"

In the infinite sequence of integers, any integer's two immediate neighbors will have the **same parity** (both odd or both even and different from that of the integer between them) and will always have a **difference of 2**. It is therefore trivial that the statement "*26 is the only integer directly sandwiched between perfect powers*" can be reinstated as (is congruent to): "***in the infinite sequence of perfect powers a^n where a,n >1, there is a singular incidence of 2 being the difference between any consecutive (adjacent) perfect powers".***

2. Divisibility and Isolation by Parity

The difference between any pair of consecutive perfect powers of uniform parity and polarity (both positive and either both odd or both even, with no perfect power of different parity in between them) will always be divisible by 4 either without remainder or with a remainder of 2:

- **even perfect powers** expressed as
$(2x)^n$, where x≥1 and n≥2,
will always have a difference of at least
$2^2=4$
because
$(2x)^n=(2^n)(x^n)$ has a minimum value of
$(2^2)(1^2)=4$
and as values of x and n change (increase)
$(2^n)(x^n)$ will always evaluate to some nonzero (x^n) multiple of (2^n) which is always a multiple of 4,
thus their differences will indeed always be divisible by 4 without remainder:
$(2^{n+p})(x^{n+q})-(2^n)(x^n)\equiv 0 \bmod 4$,
for which 2 is not a valid value
therefore 2 cannot be an element in the infinite sequence of first differences of consecutive even perfect powers.
Hence, <u>there exist no integers that are neighbored on both sides by even perfect powers!</u>

By a similar token:

- **odd perfect powers** expressed as
$(2x+1)^n$, where x≥1 and n≥2,
can be expanded to a simplified formula of the Binomial Theorem
$\Sigma(k=0, n)\ C(n\ k)\ (2x)^k$
where $(2x)^k$ will always produce values that either contain 2, as in the minimal case
$(2\times1)^1$ when x=1, n=2,
or
contain 2 + some nonzero multiple of 2 when x and n change (increase), since the lower bound value of exponent k is not tied to the value of n (only its upper bound is when, k=n) and exponent k will always take the value 1, effectively producing a nonzero multiple of 2 in $(2x)^k$ as part of the added terms determined by the coefficients C(n k) in the binomial expansion (see Pascal's Triangle);

moreover,

the two unit coefficients (1's), first and last terms in the expanded binomial summation expression of odd perfect powers, will always add 1 (ensuring odd parity) and some multiple of 4, respectively,

while
inner coefficients of varying parity in the symmetry (i.e. those between the 1's) will result in further adding both products and exponentials (≥2) of 2 to values of sequence elements,

THEREFORE,

subtracting such an odd perfect power from the larger consecutive (adjacent) one results in an even number (the 1's 'cancel out') that is **<u>either 2</u>** (the smallest element and conjected to be a singular incidence) **<u>or</u>** **<u>2 plus a multiple of 4</u>,**

CONFIRMING

the stated divisibility property also for the first differences of consecutive **odd** perfect powers (see sequence below). We notice that as bases and exponents increase, generally but not necessarily so do the differences, even decreasing, albeit rarely. The lowest value based on extensive computation to which these differences fall back ('fallback') is 8 in the even cases and **10** ("2^2×2+2", see 'multimorphism' below) in the more relevant odd cases, therefore **I conjecture that 2 is a singular incidence and 26 is the only number directly neighbored by perfect powers!** As part of our informal collaboration, a computer code written and run in 2024 by William Heyman (USA) in the range 1-1558! (an integer made up of 4,300 digits) confirmed this result.

Unlike all other, larger elements in the infinite sequence of first differences of odd consecutive perfect powers, the smallest value **<u>2 is the only one producing a zero quotient when divided by 4</u>**, in other words, the *value 2* presents the *singular incidence* of an element ***not*** *coefficially divisible* by either 4, or by 4 with a remainder of 2, a mathematical exception, if you like, of "*exceptio probat regulam".*

Although we did not initially specify that such a *coefficial divisibility* constraint must be satisfied, the value of such *singular incidence* clearly highlights the validity of such a lemma, one that is, on the one hand, different from a formal mathematical concept of divisibility, yet one that may be fully described and defined with sound mathematical formality and common understanding, as follows:

$\lfloor a/b \rfloor \geq 1$, where $a,b \in Z+$
and
$a=bq+r$ for integers $q\geq1$, $0\leq r<b$, where $r \in R$

and $R \subseteq \{0,1,...,b-1\}$ is a specified set of allowable remainders associated with the **<u>coefficial divisibility</u>** context. The pair (q,r) satisfies the condition of this *meaningful division*, implying that the division produces a shareable portion $q\geq1$ and a remainder that is not excluded by the definition.

Let $R=\{0,2\}$

An integer a is *coefficially divisible by 4* if:

$\lfloor a4 \rfloor \geq 1$ and $r \in \{0,2\}$

So, e.g.:

- 8÷4=2 with r=0: valid
- 10÷4=2 with r=2: valid
- 2÷4=0: invalid (quotient is 0)
- 3÷4=0: invalid (quotient is 0)

This, perhaps novel concept of *coefficial divisibility* aims to refine classical or formal definitions of divisibility by excluding trivial or *non-meaningful* results (e.g. “0 shares”) and introduces both cognitively and linguistically resonant conditions — only results where division implies *meaningful allocation* are accepted as valid (*‘unlike four or more, merely two pebbles cannot be divided and shared fairly among four boys for water skimming’*).

Integer sequence of the **first difference of consecutive odd perfect powers** up to the first 1,000 perfect powers (sequence 1):

2, 4, 18, 18, 38, 10, 12, 100, 106, 128, 30, 148, 154, 166, 198, 260, 138, 216, 26, 28, 270, 248, 284, 546, 524, 506, 574, 652, 250, 726, 94, 568, 170, 548, 954, 1000, 638, 180, 890, 412, 888, 674, 908, 1170, 1262, 1036, 782, 1048, 1590, 734, 252, …, ad infinitum

The two elements before reaching the 10,000th perfect power are: (90,224,199-90,231,001)=**6,802** and (90,518,849-90,535,225)=**16,376**

For the sake of balance and completeness, here is the integer sequence of the first difference of consecutive **even** perfect powers, too, listed also up to the first 1,000 perfect powers (sequence 2):

4, 4, 16, 20, 28, 24, 36, 68, 40, 56, 32, 100, 92, 168, 100, 152, 48, 104, 356, 100, 116, 496, 104, 80, 8, 144, 368, 316, 28, 732, 648, 784, 252, 704, 184, 828, 964, 200, 832, 396, 1148, 728, 732, 1208, 656, 112, 932, 804, 196, 748, 804, 316, 764,...

All elements in a combined and ordered infinite integer sequence of the first differences of consecutive perfect powers of uniform parities are “coefficially divisible” (quotient >0) by either 4 without remainder or by 4 with a remainder of 2, <u>except</u> in the case of the smallest, first (and perhaps only) element value that is <u>2</u>, presenting a singular incidence which corresponds perfectly to both the only singular incidence of an integer in Pascal’s Triangle (2) and to the singular incidence of integer ‘multimorphism’ (see below, under 4.), i.e. integer 2.

Also, the stationary adjacency a(n)=a(n+1) among the infinite number of terms of the above two integer sequences is here conjectured to occur exactly once. This uniqueness, or singularity may be explained by the unique multimorphism of integer 2, as the repeated even-case value **4** arises from 2x/+2=2^2, while the odd-case value **18** from (2x/+2)^2+2, reflecting that **<u>2</u>** is the only uniquely occurring integer (binomial coefficient) in Pascal’s Triangle.

3. Conclusion

We can conclude that the only known case (or singular incidence) when an element in the combined and ordered infinite sequence of the first differences of consecutive perfect powers of uniform parities takes the value 2 is at the very beginning, being the result of the case when base and exponent variables are lowest and one odd perfect power is followed by another, with no even perfect power between them, and one element (one side of the inequality) is, by definition of the polarity-part of the conjecture, too, larger than the other, namely, when

$(2x+1)^m > (2y+1)^n$, where $x,y \geq 1$ and $m,n \geq 2$,
which occurs when
$(2\times1+1)^{\mathbf{3}}=27 > (2\times2+1)^{\mathbf{2}}=25$
and 27-25=2.
Note that, although $(2\times1+1)^{\mathbf{2}}=9$ and $(2\times1+1)^{\mathbf{3}}=27$ do satisfy the established *inequality constraint*, $(2\times2)^{\mathbf{2}}=16$ is between them and thus they are NOT *consecutive* (adjacent) odd perfect powers without an even perfect power between them.

4. Singularities of integer 2 and Multimorphism

In addition to the invariant 2 in the Euler formula (a.k.a. Euler-Poincaré characteristic) for simple polyhedra, the singular incidence of 2 among the *coefficients* encoded in the combination multiplier (“k out of n”) within the expanded binomial theorem (listed in rows of ‘Pascal's Triangle’) corresponds perfectly with both its singularity as the ***‘multimorphic integer***’ whose multiplication, addition and exponentiation with itself yield equal results ($2+2=2\times2=2^2$) and with the singular incidence of 2 being the difference between any consecutive perfect powers which is, as shown above, congruent to stating that 26 is the only positive integer ‘sandwiched between’ two perfect powers, ***a square and a cube*** — which is why I call 26 the “Transdimensional Number’.

Interestingly, the singular incidences of the unique **multimorphic constant** (4) and the unique **multimorphic integer** (2) happen to be the very ***divisor*** and ***remainder*** values discussed in the divisibility lemma at hand.

DEFINITIONS OF MULTIMORPHISM

An integer *a* is said to be **multimorphic** if the operations of **addition**, **multiplication**, and **exponentiation applied to *a* with itself all yield the same result**:
$a+a=a\times a=a^a$
The only integer satisfying this condition is 2, when:
$2+2=4$, $2\times2=4$, $2^2=4$

Thus, **2 is the unique multimorphic integer**.

The **multimorphic constant** is the singular value that results from the triadic self-operation (addition, multiplication, exponentiation) of the multimorphic integer, i.e. the common output of the expressions:

$a+a$, $a\times a$, a^a

When a=2, all three expressions evaluate to 4. Hence, **4 is the multimorphic constant**.

5. Analogy

Broadly speaking, coefficial divisibility is kind of analogous to looking at a traditionally built brick wall where one row starts with a full brick and its adjacent row starts with a half-brick: the row starting with a full brick can be viewed as the sequence of first differences between consecutive even perfect powers (marked as ΔCEPP on my illustration) and the row starting with a half brick can be viewed as the sequence of first differences of consecutive odd perfect powers (“ΔCOPP”). If we picture a chimney or a well instead of a wall, a vertical line collinear with the edge of a full brick would represent the starting point, or zero.

Without proper tools or adequate methods and conditions to properly divide (cut/break) a brick, no wall starting (and/or ending) in a straight line could be built using the traditional placement of bricks proven to ensure optimal stability. The condition of “coefficial divisibility” may just present us with such a useful tool of division, particularly so with respect to mathematical proofs in number theory. After all, that cut-off half-brick will come real handy, if the row that begins with its other half is ever to be completed.

6. Coefficial Divisibility and Pillai's Conjecture

The above ideas highlight the validity of this novel approach to integer divisibility described by the simple rule of defining 1 as the minimal value for a quotient (invalidating zero) — a method that can be closely aligned with both human cognitive development and formal mathematical thought.

In addition to describing the exceptional property of the conjectured singular incidence of value 2 among the differences between consecutive perfect powers (see Catalan's Conjecture / Mihailescu-theorem for the case of 1), the methods used may open the way for further proofs and especially for simplifying existing proofs, or attempts thereof, related to the finite incidences of integers other than 2 in Pillai's Conjecture.

In memory of
Subbayya Sivasankaranarayana Pillai
(5 April 1901 – 31 August 1950)

17 April 2025

References:

Catalan’s Conjecture
- Weinstein, E.W.: Catalan’s conjecture https://mathworld.wolfram.com/CatalansConjecture.html
- First Differences of Consecutive Perfect Powers, Online Encyclopedia of Integer Sequences A053289 https://oeis.org/A053289

Pillai’s Conjecture
- Online Encyclopedia of Integer Sequences A001579 https://oeis.org/A001597
- Wikipedia https://en.wikipedia.org/wiki/Catalan%27s_conjecture
- Ghannouchi, J., A proof of Pillai conjecture. 2012. hal-00698687v22
- Bennett and Siksek, Differences between perfect powers: prime power gaps, Algebra and Number Theory Vol. 17, 2023/10, DOI 10.2140/ant.2023.17.1789